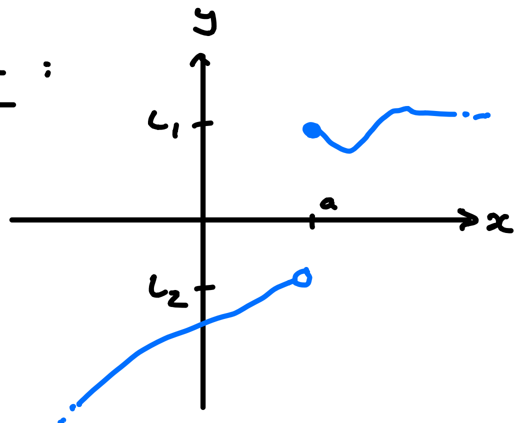
Failure of Limits to Exist

$$\left.\begin{array}{l}\lim\limits_{x \to a^+} f(x) = L_1 \\ \qquad\qquad\qquad\quad \neq \\ \lim\limits_{x \to a^-} f(x) = L_2\end{array}\right. \implies \lim_{x \to a} f(x) \text{ DNE}$$

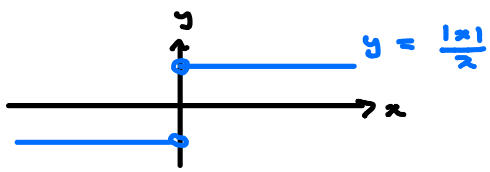
Basic Picture:



Warning: $f(a) = L_1$ is irrelevant. The limit does not care about $f(a)$.

Example 1/ Find $\displaystyle\lim_{x \to 0} \frac{|x|}{x}$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \implies \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



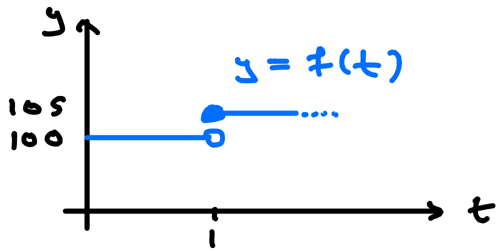
$$\implies \begin{array}{l} \lim\limits_{x \to 0^+} \frac{|x|}{x} = 1 \\ \lim\limits_{x \to 0^-} \frac{|x|}{x} = -1 \\ \lim\limits_{x \to 0} \frac{|x|}{x} \text{ DNE} \end{array}$$

2/ (Annual compound interest)

$P = 100$

$r = 0.05$

$f(t) =$ Balance at time t in years after initial deposit.

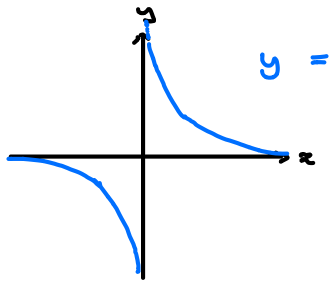


$$\Rightarrow \lim_{t \to 1^-} f(t) = 100$$

$$\lim_{t \to 1^+} f(t) = 105$$

$$\lim_{t \to 1} f(t) \text{ DNE}$$

Q: What is $\lim_{x \to 0^+ / 0^- / 0} \frac{1}{x} = ?$



$$\Rightarrow \lim_{x \to 0^+, 0^-, 0} \frac{1}{x} \text{ DNE.}$$

Definition

$\lim_{x \to a^+ / a^- / a} f(x) = \infty \iff f(x)$ grows positively without bound as x approaches (but does not equal) a from above / below / both sides

Just notation.

$\lim_{x \to a^+ / a^- / a} f(x) = -\infty \iff f(x)$ grows negatively without bound as x approaches (but does not equal) a from above / below / both sides

Warning: Despite the notation the limit DNE in both cases. A limit is always a number. $\pm\infty$ is not a number.

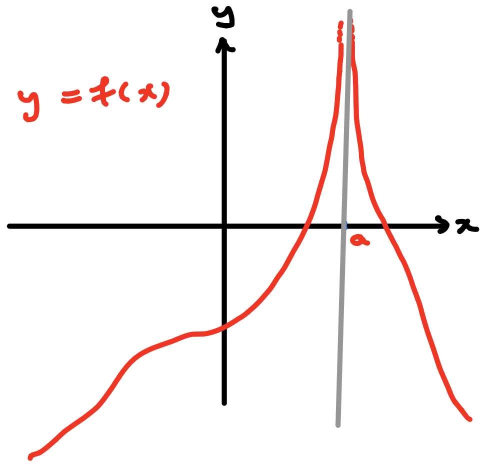
Limit Exists

DNE ∞ $-\infty$

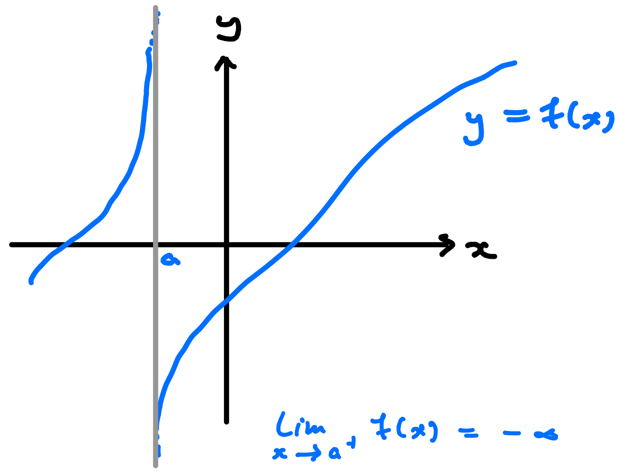
Example: $\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ DNE and is neither $\pm\infty$.

Basic Graphs :



$$\lim_{x \to a^+ / a^- / a} f(x) = \infty$$



$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a} f(x) \neq \pm\infty$$

In any of these cases, we say the vertical line $x = a$ is a <u>vertical asymptote.</u>

<u>Facts</u> (Also work for $x \to a^+$ and $x \to a^-$)

1/ $\lim\limits_{x \to a} g(x) = \pm\infty \Rightarrow \lim\limits_{x \to a} \dfrac{1}{g(x)} = 0$

2/ $\left.\begin{array}{l} \lim\limits_{x \to a} f(x) = A > 0 \\ \lim\limits_{x \to a} g(x) = 0^+ \end{array}\right\} \Rightarrow \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \infty$

↖ approaches 0 positively

3/ $\left.\begin{array}{l} \lim\limits_{x \to a} f(x) = A > 0 \\ \lim\limits_{x \to a} g(x) = 0^- \end{array}\right\} \Rightarrow \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = -\infty$

↖ approaches 0 negatively

Example $\lim\limits_{x \to 1^+/1^-/1} \dfrac{x^2-2x+1}{(x-1)^3} = ?$

$x^2-2x+1 = (x-1)^2 \Rightarrow \dfrac{x^2-2x+1}{(x-1)^3} \overset{\forall\, x \neq 1}{=} \dfrac{1}{x-1}$

$\Rightarrow \lim\limits_{x \to 1^+/1^-/1} \dfrac{x^2-2x+1}{(x-1)^3} = \lim\limits_{x \to 1^+/1^-/1} \dfrac{1}{x-1}$

$\lim\limits_{x \to 1^+} x-1 = 0^+ \Rightarrow \lim\limits_{x \to 1^+} \dfrac{1}{x-1} = \infty$

$\lim\limits_{x \to 1^-} x-1 = 0^- \Rightarrow \lim\limits_{x \to 1^-} \dfrac{1}{x-1} = -\infty$

$\lim\limits_{x \to 1} \dfrac{1}{x-1}$ DNE (and is neither $\pm\infty$)

Example $\lim\limits_{x \to 4^+} \dfrac{\sqrt{x}(x-4)}{x^2-8x+16} = ?$

$\lim\limits_{x \to 4^+} \sqrt{x}(x-4) = \sqrt{4}(4-4) = 0$

$\lim\limits_{x \to 4^+} x^2-8x+16 = 4^2-8\cdot 4+16 = 0$

$\Rightarrow$ Must simplify using algebra

$\lim\limits_{x \to 4^+} \dfrac{\sqrt{x}(x-4)}{x^2-8x+16} = \lim\limits_{x \to 4^+} \dfrac{\sqrt{x}(x-4)}{(x-4)^2} = \lim\limits_{x \to 4^+} \dfrac{\sqrt{x}}{x-4}$

$$\left.\begin{aligned} &\lim_{x \to 4^+} \sqrt{x} = \sqrt{4} = 2 > 0 \\ &\lim_{x \to 4^+} x-4 = 0^+ \end{aligned}\right\} \Rightarrow \lim_{x \to 4^+} \frac{\sqrt{x}}{x-4} = \infty$$